ON THE SIZE OF THE SET OF POINTS WHERE THE METRIC PROJECTION EXISTS

BY

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ABSTRACT

In this paper we answer in the negative a question due to J. P. R. Christensen about almost everywhere existence of nearest points using a decomposition of ℓ_2 due to J. Matoušek and E. Matoušková. We also formulate a similar question about almost everywhere existence of farthest points and answer it in the negative.

1. Introduction

Let X be a (real) Banach space, let F be a closed subset of X. We define the **distance function**

$$f(x) = \operatorname{dist}(x, F) = \inf\{\|y - x\| \colon y \in F\}$$

and the **metric projection**

$$P_F(x) = \{y \in F \colon \|y - x\| = f(x)\},$$

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which assigns to each $x \in X$ the set of **nearest points** in F to x. We say that P_F is **continuous** at x provided $P_F(x)$ is single-valued and $y_n \to P_F(x)$ whenever $x_n \to x$ and $y_n \in P_F(x_n)$. We also define the **directional derivative**

$$r_f(x,v) = \lim_{t\to 0} t^{-1}(f(x+tv) - f(x)) \quad (= D_v f(x))$$

for $0 \neq v \in X$, if the limit exists. The metric projection was studied by many authors — see for example [F1, F2, Z1, Z2]. A nice survey about existence of nearest points appeared in [BF]. It is natural to ask for how many points of $X \setminus F$ do there exist closest points in F. The following theorem due to Lau gives an answer to this question in terms of Baire category:

THEOREM 1.1 (Lau, see [BF]): *If E is a reflexive Kadec*[1] *space, then for each closed nonempty set C in E the set of points of $E \setminus C$ with nearest points in C contains a dense G_δ subset of $E \setminus C$.*

We will prove a theorem which resolves in the negative the two following conjectures by J. P. R. Christensen [Ch], which are concerned with the existence of nearest points in terms of measure. Let X be a separable Banach space. Then:

(I) The differential $r_f(x,\cdot)$ is for almost every $x \in X \setminus F$ an element of the dual space X^* of norm one and for almost every $x \in F$ we have $r_f(x,\cdot) = 0$. It seems most likely that this is true also in the non-reflexive case.

(II) If X is reflexive, then for almost every $x \in X$ there exist $y \in F$ with $\|x - y\| = f(x)$ and y is unique for almost every $x \in X$ if X is strictly convex.

In the present context, "almost every" means outside of a Haar null set (see definition below). The following theorem of S. Fitzpatrick implies (given the assumptions are satisfied) that[2] if $\|r_f(x,\cdot)\|_{X^*} = 1$, then the metric projection P_F is continuous at x (and thus $P_F(x) \neq \emptyset$).

THEOREM 1.2 (S. Fitzpatrick, Corollary 3.4, [F1]): *Suppose that the norms of X and X^* are Fréchet differentiable. If $r_f(x,u) = 1$ for some $u \in S(X)$, then P_F is continuous at x.*

Here $S(X)$ denotes the unit sphere of X. Our example relies heavily on a construction of E. Matoušková from [Mat], which is a generalization of the previous result of J. Matoušek and E. Matoušková from [MM] to superreflexive spaces and

1 For the definition of Kadec spaces, see [BF].

2 If the norm of X^* is Fréchet differentiable, then it follows that X is reflexive — see for example [DGZ].

gives an explicit construction of an equivalent norm, which is almost nowhere (in the sense of Aronszajn) Fréchet differentiable. The construction supplies a Borel set $D \subset \ell_2$ which is ball small and whose complement is Aronszajn null. Now an application of our Proposition 3.2 yields a Borel set $Q \subset \ell_2$, which is not Haar null, and a nonempty closed set $A \subset \ell_2$ so that $P_A(x) = \emptyset$ for all $x \in Q$. An application of Fitzpatrick's Theorem 1.2 yields a counterexample to conjectures (I) and (II) of Christensen.

Let M be a closed bounded nonempty subset of a Banach space X. For $x \in X$ we shall define the **farthest distance to x in M** as

$$\rho(x, M) = \sup\{\|x - f\|: f \in M\}.$$

We also define the **farthest point map** as

$$F_M(x) = \{f \in M: \|x - f\| = \rho(x, M)\}.$$

The following theorem is due to E. Asplund

THEOREM 1.1 (Asplund, see [A]): *Let X be a reflexive space with a LUR[3] norm and $M \subset X$ be a closed bounded nonempty subset. Then the set of points $x \in X$ with $F_M(x) \neq \emptyset$ is a dense G_δ subset of X.*

In analogy with Christensen's conjecture, we can ask whether farthest points to a closed bounded set, say in a Hilbert space, exist almost everywhere in the sense of Christensen (i.e., outside of a Haar null set). Again, using the construction due to E. Matoušková (see [Mat]), we construct a closed convex bounded nonempty set $M \subset \ell_2$ with the property that the set of points $x \in \ell_2$, where $F_M(x) = \emptyset$, contains a Borel subset, which is not Haar null — see Theorem 4.3.

Finally, let C be a nonempty weakly compact subset of X. Let us recall the following theorem due to Lau [La], which tells us that farthest points to a weakly compact set exist "almost everywhere" in terms of category.

THEOREM 1.4 (Lau, see [DGZ], Proposition 2.7): *Let X be a Banach space, and let C be a nonempty weakly compact subset of X. Then there exists G, which is a dense G_δ subset of X, and if $x \in G$, then x has a farthest point in C.*

Our Theorem 4.3 shows[4] that an analogy of this theorem is not possible, if we consider "almost everywhere" in terms of measure (Haar null sets acting as negligible sets).

3 For the definition of *LUR*, see [DGZ]. Let us only remark that if a norm is *LUR*, then it follows that the space is Kadec.

4 It is an easy consequence of reflexivity and Mazur's theorem that a closed convex bounded subset of ℓ_2 is weakly compact.

2. Preliminaries

Notation 2.1: Let X be a normed linear space, let $x \in X$ and $r > 0$. By $B(x,r)$ (respectively by $\overline{B}(x,r)$) we shall denote the open (respectively closed) ball with center x and radius r. By $S(x,r)$ we shall denote the sphere $S(x,r) = \{x \in X\colon \|x\| = r\}$, and by S_X the sphere $S(0,1)$. By $\overline{A}$ we shall denote the closure of A.

Definition 2.2: Let X be a separable Banach space. Let A be a Borel subset of X. The set A is called **Haar null** if there is a Borel probability measure μ on X such that $\mu(x + A) = 0$ for every $x \in X$.

Let $B \subset X$ be Borel. We say that B is **Aronszajn null** if for every sequence $(x_i)_{i=1}^{\infty}$ in X whose closed linear span is X there exist Borel sets $B_i \subset X$ such that $B \subset \bigcup_i B_i$ and the intersection of B_i with any line in the direction x_i has the one-dimensional Lebesgue measure zero, for each $i \in \mathbb{N}$.

Aronszajn null sets are Haar null but the converse is not true. For more information about Haar and Aronszajn null sets see [BL]. We only note that a Borel set $A \subset X$ is Haar null if and only if λA is Haar null for every $\lambda > 0$. The following notion was introduced by D. Preiss and L. Zajíček in [PZ]:

Definition 2.3 ([PZ]): Let X be a normed linear space and $A \subset X$, $r > 0$. We say that A is **r-ball porous** if for each $x \in A$ and $\varepsilon > 0$ there exists $y \in X$ such that $\|x - y\| = r$ and $B(y, r - \varepsilon) \cap A = \emptyset$. We say that $B \subset X$ is **ball small**, if $B = \bigcup_n A_n$ such that A_n is r_n-ball porous where $r_n > 0$.

Remark 2.4:

(1) It is easily seen that if A is a Borel ball small set, then A_n can also be chosen Borel.

(2) If X is a finite dimensional Banach spaces, then by compactness we can take $\varepsilon = 0$ in the definition of r-ball porosity.

(3) It is easy to see that r-ball porosity of a set A follows from the condition: for each $x \in A$ there exist $y_j \in X$ such that $B(y_j, r) \cap A = \emptyset$ and $\operatorname{dist}(x, B(y_j, r)) \to 0$ as $j \to \infty$.

(4) Note that if A is r_0-ball porous at some $x \in A$, then it is also r-ball porous at x for any $0 < r < r_0$.

Let us note that "being a ball small set" is a metric property. The following example shows that it indeed depends on the particular norm.

Example 2.5: Let $n \in \mathbb{N}, n \geq 2$. Then there exists a set $A \subset \mathbb{R}^n$, which is not ball small for the Euclidean norm and there exists a norm $|\cdot|_b$ on $\mathbb{R}^n$ for which it is 1-ball porous (and thus ball small).

Proof: According to [PZ] there exists $A \subset \mathbb{R}^n$, which is a subset of a Lipschitz (even delta-convex) hypersurface (i.e., there exists a subspace $Y \subset \mathbb{R}^n$, $\dim(Y) = n-1$, $0 \neq v \in \mathbb{R}^n$, and a delta-convex function $f: Y \to \mathbb{R}$, such that $A \subset \{y + f(y) \cdot v: y \in Y\}$), such that A is not ball small for the Euclidean norm. Delta-convex functions are those functions that are representable as a difference of two convex functions (see [DVZ]). We can assume that $Y = \mathbb{R}^{n-1}$ and $v = (0, \dots, 0, 1)$.

Now it's enough to construct a norm $|\cdot|_b$, in which A is ball small. The fact that f is Lipschitz implies that there exists a cone

$$K = \{(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R}: \|x\| \leq Ct\},$$

where $C > 0$, such that $(a + K) \cap A = \{a\}$ for each $a \in A$. It is readily seen, that there are $\alpha_i > 0$ for $i = 1, \dots, n$, such that $|(x_1, \dots, x_n)|_b := \sum_i \alpha_i |x_i|$ has the following property:

$$0 \in \overline{B}_{|\cdot|_b}((0, \dots, 0, 1/\alpha_n), 1) \subset K.$$

Now it is easy to see that A is 1-ball porous for $|\cdot|_b$. ∎

Definition 2.6: Let X be a Banach space. If $0 \neq v \in X$ and $0 < c < \|v\|$, we define **the cone**

$$A(v,c) = \{x \in X: x = \lambda v + w, \lambda > 0, \|w\| < c\lambda\} = \bigcup_{\lambda > 0} \lambda B(v,c).$$

Now let $M \subset X$ and $x \in M$. We say that M is **cone-supported at** x if there exists a cone $A(v,c)$ and $r > 0$ such that

$$M \cap (x + A(v,c)) \cap B(x,r) = \emptyset.$$

A set is called **cone-supported** if it is cone-supported at all its points. A set is called **σ-cone-supported** if it can be written as a union of countably many cone-supported sets.

Let $r > 0, s \in X$. We say that M **is supported at** x **by** $B(s,r)$, if $\|s - x\| = r$ and $B(s,r) \cap M = \emptyset$.

We will need the following lemmas. The proof of the next statement is simple and so we omit it.

LEMMA 2.7: *Let X be a Banach space and let $C \subset X$. If for some $x \in C$ there exist $r > 0$ and $y \in X$ such that $\|x - y\| = r$, and $B(y,r) \cap C = \emptyset$ (i.e., C is supported at x by $B(y,r)$), then C is cone-supported at x.*

Definition 2.8: Let X be a Banach space. We say that $A \subset X$ is a **Lipschitz hypersurface** if there exists a subspace $Y \subset X$ such that $X = Y \oplus \mathbb{R}v$ (topologically), where $0 \neq v \in X$ and there exists a Lipschitz function $g: Y \to \mathbb{R}$ such that $A = \{y + g(y) \cdot v: y \in Y\}$.

LEMMA 2.9 ([Z2]): *Let X be a separable Banach space. Then a set $M \subset X$ is σ-cone-supported if and only if it can be covered by countably many Lipschitz hypersurfaces.*

(For a proof, see for example [Z2].)

LEMMA 2.10: *Let $z, f \in S_{\ell_2}$, $\langle z, f\rangle = 0$, and let $R_0 > 0$. Let us define $v = z + R_0 f$.*

(i) *We have the following:*

$$B(v, R_0) \subset C_v = \{x \in \ell_2: \langle v, x\rangle > \|x\|\},$$

$B(v/\|v\|, R_0/\|v\|) \subset C_v$, *and* $z \in \overline{B}(v, R_0)$.

(ii) *Let $0 < R < R_0$ and $s = z + Rf$. Then $B(s/\|s\|, R/\|s\|) \subset C_v$, $z \in \overline{B}(s, R)$, and*

$$\operatorname{dist}(s/\|s\|, z) = (2(1 - (1 + R^2)^{-1/2}))^{1/2}.$$

Proof: Most of these facts follow by easy computations. The only fact which is not completely obvious is the inclusion $B(s/\|s\|, R/\|s\|) \subset C_v$. To see this, note that C_v is a cone and so $\lambda C_v \subset C_v$ for any $\lambda > 0$. Take $\lambda = 1/\|v\|$ and the inclusion $B(v/\|v\|, R_0/\|v\|) \subset C_v$ follows. Because $B(s, R) \subset B(v, R_0)$, by the same scaling argument with $\lambda = 1/\|s\|$, we obtain that $B(s/\|s\|, R/\|s\|) \subset C_v$. ∎

Our methods rely upon the following construction due to E. Matoušková from [Mat]. Let $(e_k)_{k \in \mathbb{N}}$ be the standard basis in ℓ_2. For $x \in \ell_2$, let $\operatorname{supp} x = \{i \in \mathbb{N}: \langle e_i, x\rangle \neq 0\}$. Let $(x_k)_{k \in \mathbb{N}}$ be a dense subset of S_{ℓ_2} with the property that each x_k is finitely supported. Let $\varepsilon > 0$ and let $0 < n_1 < n_2 < \cdots$ be a sequence in $\mathbb{N}$ satisfying $n_k > \max \operatorname{supp} x_k$. Then define $v_k := x_k + r e_{n_k}$, where $r = r(\varepsilon) > 0$ is chosen to be small enough (see Proposition 2.11). We get that $\|v_k\| = (1 + r^2)^{1/2}$. Put

$$\widetilde{S}_\varepsilon := \bigcup_{k \in \mathbb{N}} \{x \in \ell_2: \langle v_k, x\rangle > \|x\|\} \quad \text{and} \quad S_\varepsilon = \widetilde{S}_{\varepsilon/2} \cup -\widetilde{S}_{\varepsilon/2}. \tag{2.1}$$

Then S_ε is an open subset of ℓ_2. We need the following version of Proposition 2.5 of [Mat] for the case of ℓ_2.

PROPOSITION 2.11 (Proposition 2.5 of [Mat] for ℓ_2): *There exists $N \in \mathbb{N}$ with the following property: for each $\varepsilon > 0$ there exists $0 < r < 2$ such that*

(1) $\overline{S}_\varepsilon = \ell_2$, *and*

(2) $\lambda_Z(S_\varepsilon \cap B_Z) \leq \varepsilon$ *for each N-dimensional subspace $Z \subset \ell_2$,*

where λ_Z is the Lebesgue measure on the (Euclidean) subspace $Z \subset \ell_2$.

The following proposition is a consequence of E. Matoušková's construction.

PROPOSITION 2.12:

(i) *There exists a Borel set $D \subset \ell_2$, which is ball small, and whose complement is Aronszajn null.*

(ii) *There exist a closed nonempty set $E \subset \ell_2$, and $R_0 > 0$ such that for each $0 < R \leq R_0$ we have*

  (1) *E is not Haar null,*

  (2) $E \subset \overline{B}_{\ell_2}(0,1) \setminus B_{\ell_2}(0,1/2)$, *and $E = -E$,*

  (3) *E is "radial" in the following sense: if $x \in E$, then E contains the closed line segment joining $x/(2\|x\|)$ and $x/\|x\|$,*

  (4) *for each $x \in E \cap S(0,1)$ the following is true:*

  (*) *there exists a sequence $u_j = u_j(x) \in S(0,1)$, such that*

  – $\|x - u_j\| \to (2(1-(1+R^2)^{-1/2}))^{1/2}$ *as $j \to \infty$,*

  – *if $R < \sqrt{3}$, then $\operatorname{dist}(u_j, E) \to R/(1+R^2)^{1/2}$ as $j \to \infty$, and*

  – $B(u_j, R/(1+R^2)^{1/2}) \cap \bigcup_{\lambda>0} \lambda E = \emptyset$,

  for each $j \in \mathbb{N}$.

Proof: If we apply Proposition 2.11 for $\varepsilon = 1/n$, then we obtain sets $S_{1/n}$ and $r_n > 0$, such that $F = \bigcap_n S_{1/n}$ is Aronszajn null. This is a consequence of Lemma 2.2 from [Mat], because by property (2) of Proposition 2.11 we have that $\lambda(F \cap B_Z) \leq \varepsilon$ for any N-dimensional subspace $Z \subset \ell_2$ and for any $\varepsilon > 0$. Put $D = \ell_2 \setminus F = \bigcup_n (\ell_2 \setminus S_{1/n})$.

Fix $n \in \mathbb{N}$ and for $m \in \mathbb{Z}$ let

$$T_m = T_m^n = (\ell_2 \setminus S_{1/n}) \cap (\overline{B}(0, 2^{m+1}) \setminus B(0, 2^m)). \tag{2.2}$$

Each T_m is closed and we have that

$$(\ell_2 \setminus S_{1/n}) \setminus \{0\} = \bigcup_m T_m.$$

We shall prove that each T_m is t-ball porous for some $t > 0$. This will conclude the proof of part (i) of our proposition, because $D = \{0\} \cup \bigcup_n \bigcup_m T_m^n$. By rescaling, it is enough to establish t-ball porosity of T_{-1} (as $T_k = \lambda_{kl} \cdot T_l$ for a suitable $\lambda_{kl} > 0$). Let $x \in T_{-1}$. Note that $\|x\| \in [1/2, 1]$. Assume first that $\|x\| = 1$. Find $x_{k_j} \in S_{\ell_2}$ so that $x_{k_j} \to x$ as $j \to \infty$ (see the text preceding (2.1) for the definition of $(x_k)_{k \in \mathbb{N}}$). Define $R_0 = r_n$ ($r_n = r(1/n)$ — see the comments preceding Proposition 2.11). By part (i) of Lemma 2.10 applied to $z = x_{k_j}$ and $f = e_{n_{k_j}}$ (and $v_{k_j} = x_{k_j} + R_0 e_{n_{k_j}}$) we get that $x_{k_j} \in \overline{B}(v_{k_j}, R_0)$ and $B(v_{k_j}, R_0) \cap T_{-1} = \emptyset$, because $B(v_{k_j}, R_0) \subset C_{v_{k_j}}$ and $C_{v_{k_j}} \cap T_{-1} = \emptyset$. This also implies that $B(v_{k_j}, R_0) \cap \bigcup_{\lambda > 0} \lambda T_{-1} = \emptyset$. Now $\operatorname{dist}(x, B(v_{k_j}, R_0)) \to 0$ as $x_{k_j} \in \overline{B}(v_{k_j}, R_0)$ and $x_{k_j} \to x$ as $j \to \infty$. Condition (3) from Remark 2.4 shows that T_{-1} is R_0-ball porous at x.

Now consider $x \in T_{-1}$ with $\|x\| < 1$. By radiality, $y := x/\|x\| \in T_{-1}$, and by the last paragraph there exist $w_j = v_{k_j}$ with $B(w_j, R_0) \cap \bigcup_{\lambda > 0} \lambda T_{-1} = \emptyset$, and $\operatorname{dist}(y, B(w_j, R_0)) \to 0$ as $j \to \infty$. Define $\tilde{w}_j := w_j/\|x\|$. Then

$$B(\tilde{w}_j, R_0/\|x\|) \cap T_{-1} = \emptyset \text{ and } \operatorname{dist}(x, B(\tilde{w}_j, R_0/\|x\|)) \to 0 \quad \text{for } j \to \infty.$$

Thus we can conclude (again from condition (3) of Remark 2.4) that T_{-1} is $R_0/\|x\|$-porous at x. Now observe that $1/2 \le \|x\| \le 1$ for $x \in T_{-1}$, and thus T_{-1} is $R_0/2$-ball porous (see condition (4) of Remark 2.4).

To prove part (ii) of our proposition, note that $\ell_2 \setminus F = \bigcup_n (\ell_2 \setminus S_{1/n})$ is a set whose complement is Aronszajn null. So there exists $n_0 \in \mathbb{N}$ such that $F_1 = \ell_2 \setminus S_{1/n_0}$ is a (closed) non-Haar null set. Write $F_1 \setminus \{0\} = \bigcup_{m \in \mathbb{Z}} T_m$ (see 2.2). There exists $m_0 \in \mathbb{Z}$ such that T_{m_0} is a (closed) set, which is not Haar null. Define $E := T_{m_0}$. By rescaling, we can again assume that $m_0 = -1$. Take $R_0 := r_{n_0}$ ($r_{n_0} = r(1/n_0)$ — see the comments preceding Proposition 2.11). Conditions (1), (2), and (3) of part (ii) of our proposition are clearly satisfied. To establish condition (4), we shall prove (*) for any $x \in E \cap S(0,1)$.

Choose $0 < R \le R_0$ and $x \in E = T_{-1}$ with $\|x\| = 1$. Find $x_{k_j} \in S_{\ell_2}$ so that $x_{k_j} \to x$ as $j \to \infty$ (see the text preceding (2.1) for the definition of $(x_k)_{k \in \mathbb{N}}$). By part (ii) of Lemma 2.10 applied to $z = x_{k_j}$ and $f = e_{n_{k_j}}$, we get sequences v_{k_j} and s_j (where $v_{k_j} = x_{k_j} + R_0 e_{n_{k_j}}$ and $s_j = x_{k_j} + R e_{n_{k_j}}$) such that $x_{k_j} \in \overline{B}(s_j, R)$ and $B(s_j, R) \cap T_{-1} = \emptyset$, because $B(s_j, R) \subset C_{v_{k_j}}$ and $C_{v_{k_j}} \cap T_{-1} = \emptyset$. Define $u_j = s_j/\|s_j\|$. The first condition of (*) follows by the triangle inequality from the fact that by part (ii) of Lemma 2.10, we have that

$$\operatorname{dist}(u_j, x_{k_j}) = (2(1 - (1 + R^2)^{-1/2}))^{1/2},$$

and $\|x_{k_j} - x\| \to 0$ as $j \to \infty$. The second condition follows from the facts that

$$B(u_j, R/(1+R^2)^{1/2}) \cap E = \emptyset,$$
$$(1+R^2)^{-1/2} x_{k_j} \in \overline{B}(u_j, R/(1+R^2)^{1/2}), \quad \text{and}$$
$$(1+R^2)^{-1/2} x_{k_j} \to (1+R^2)^{-1/2} x \in E.$$

Finally, the third condition of (*) follows from the radiality of $C_{v_{k_j}}$, as

$$B(u_j, R/(1+R^2)^{1/2}) \subset C_{v_{k_j}},$$

and $C_{v_{k_j}} \cap E = \emptyset$. ∎

3. The nearest points

First, let us prove the following

PROPOSITION 3.1: *Let X be a separable Banach space, let $C \subset X$ be r-ball porous for some $r > 0$. Then there exists a nonempty closed set A and Lipschitz hypersurfaces $L_n, n \in \mathbb{N}$, such that for each $x \in \overline{C} \setminus \bigcup_n L_n$ we have that $P_A(x) = \emptyset$ (i.e., x has no nearest point in A).*

Proof: Without any loss of generality we can suppose that C is closed (r-ball porosity is stable with respect to taking closures). Define

$$A := \{x \in X \colon \operatorname{dist}(x, C) \geq r/2\}.$$

Then A is obviously closed and nonempty (by porosity).

CLAIM: $\operatorname{dist}(x, A) = r/2$ *for* $x \in C$.

Pick $x \in C$. Obviously, $\operatorname{dist}(x, A) \geq r/2$.

Now, there exists a sequence of y_n such that $\overline{B}(y_n, r - 1/n) \cap C = \emptyset$ and $\|y_n - x\| = r$. Define

$$z_n := (1/2 + 1/(nr))(y_n - x) + x \quad \text{for } n > 2/r.$$

The inclusion $B(z_n, r/2) \subset B(y_n, r - 1/n)$ implies that $z_n \in A$. It is immediate that $\|z_n - x\| \to r/2$ as $n \to \infty$. So we can conclude that $\operatorname{dist}(x, A) = r/2$.

Suppose $P_A(x) \neq \emptyset$ for some $x \in C$. Then there exists $y \in A$ which is the nearest point to x in A (i.e., $\|y - x\| = r/2$). By definition of A we get that $B(y, r/2) \cap C = \emptyset$, so by Lemma 2.7 the set

$$F = \{x \in C \colon P_A(x) \neq \emptyset\}$$

is cone supported. By Lemma 2.9, the set F can be covered by countably many Lipschitz hypersurfaces L_n. ∎

PROPOSITION 3.2: *Let X be a separable Banach space and $D \subset X$ be a Borel ball small set. Suppose that $X \setminus D$ is Aronszajn null. Then there exists a nonempty closed set A and a Borel set Q which is not Haar null such that the metric projection $P_A(x)$ is empty for each $x \in Q$.*

Proof: Let $D = \bigcup_n D_n$, where D_n is Borel r_n-ball porous for some $r_n > 0$. Then there is n_0 such that D_{n_0} is not Haar null. Put $C = D_{n_0}$ and $r = r_{n_0}$. Apply Proposition 3.1 to C and r. It yields a nonempty closed set A and a sequence of Lipschitz hypersurfaces L_n. As a Lipschitz hypersurface is even Aronszajn null (see [Z1], page 295), the set $Q = C \setminus \bigcup_n L_n$ satisfies the conclusion of the theorem. ∎

THEOREM 3.3: *Let H be a separable Hilbert space. Then there exist a Borel set Q, which is not Haar null, and a nonempty closed set $A \subset H$ such that the metric projection $P_A(x) = \emptyset$ for $x \in Q$.*

Proof: By part (i) of Proposition 2.12, there exists a Borel set $D \subset H$ which is ball small and whose complement is Aronszajn null. Now apply Proposition 3.2 to D. It yields a non-Haar null set Q and a nonempty closed set A with the required properties. ∎

4. The farthest points

Now we consider a similar problem for the farthest points. We shall need the following easy proposition (we include its proof as we were not able to find a reference in the literature):

PROPOSITION 4.1: *Let X be a Banach space with the following property: all points of S_X are strongly exposed[5] points of $\overline{B}_X$. Let M be a nonempty closed bounded subset of X and let us suppose that for an $x \in X$ there exists no farthest point in M (i.e., $F_M(x) = \emptyset$). Then there does not exist a farthest point to x in the set $\overline{\text{conv}} M$.*

Proof: For a contradiction, suppose that $f \in \overline{\text{conv}} M$ satisfies

$$\rho = \|x - f\| = \rho(x, M) = \rho(x, \overline{\text{conv}} M).$$

5 For the definition of a strongly exposed point and related notions see [BL], pages 104 and 108.

Then obviously $f \notin M$ and by closedness of M there exists $\varepsilon > 0$ such that $B(f,\varepsilon) \cap M = \emptyset$. By the assumption on X we have that all the points of the sphere S_X are strongly exposed points of the ball $\overline{B}_X$, so we can deduce the existence of $x^* \in X^*$ and $\alpha > 0$ such that

$$f \in \overline{B}(x,\rho) \cap \{z \in X\colon x^*(z) > \alpha) \subset B(f,\varepsilon) \tag{4.1}$$

(see [BL], pp. 104 and 108). It follows that

$$\overline{\text{conv}} M \subset \{z \in X\colon x^*(z) \leq \alpha\},$$

and the latter set does not contain f (by 4.1), which is a contradiction. ∎

Remark 4.2: It is easy to see that all the points of S_{ℓ_2} are strongly exposed points of $\overline{B}_{\ell_2}$ and thus ℓ_2 satisfies the assumption of Proposition 4.1.

THEOREM 4.3: *There exist a nonempty closed convex bounded (and thus weakly compact) set $M \subset \ell_2$ and a Borel set $A \subset \ell_2$, which is not Haar null, and such that $F_M(x) = \emptyset$, for $x \in A$.*

Proof: Let E be the set from part (ii) of Proposition 2.12. Take

$$R < \min(R_0, \sqrt{3}).$$

Then define

$$s := R/(1+R^2)^{1/2} < \sqrt{3}/2,$$

and put

$$l := (2(1-(1-s^2)^{1/2}))^{1/2} = (2(1-(1+R^2)^{-1/2}))^{1/2}.$$

Define

$$M := \{x \in S(0,1)\colon \operatorname{dist}(x,E) \geq s\},$$

and $\widetilde{A} = 3 \cdot E$. Note that $\widetilde{A}$ is not Haar null. Obviously, M is closed, bounded, and nonempty (by condition (4) of Proposition 2.12).

CLAIM: $\rho(a,M) = D := ((1+\|a\|)^2 - l^2 \cdot \|a\|)^{1/2}$ *for all* $a \in \widetilde{A}$.

Choose $a \in \widetilde{A}$ and $m \in M$. Define $b = -a/\|a\| \in E$. As $(1-s^2)^{1/2} b \in E$, we have (by an easy computation) that $\xi := \|b - m\| \geq l$. Let $\rho = \|a - m\|$. Then $\rho^2 = (1+\|a\|)^2 - \xi^2 \cdot \|a\|$. This function (i.e., ρ considered as a function of ξ) has a maximum at l on $[l, 2]$, $\rho(l) = D$, and thus $\rho(a,M) \leq D$. For the other inequality, suppose that

$$\rho(a,M) < D. \tag{4.2}$$

By condition (4) of Proposition 2.12, there exists the appropriate sequence $(s_j)_{j\in\mathbb{N}}$, with $\|s_j\| = 1$, and $\|s_j - b\| \to l$ as $j \to \infty$. It is easy to see that $\|s_j - a\| \to D$, and we have a contradiction with 4.2.

Suppose that for $a \in \widetilde{A}$, there exists a farthest point $m \in M$. Then $B(m,s) \cap E = \emptyset$ and E is supported at $-(1-s^2)^{1/2}a/\|a\| \in E$ by $B(m,s)$. This in turn implies that $\widetilde{A}$ is supported at a by

$$B(-\|a\|(1-s^2)^{-1/2}m, s\|a\|(1-s^2)^{-1/2}).$$

We have seen that if $a \in \widetilde{A}$ has a farthest point in M, then $\widetilde{A}$ is supported by a ball at a. By Lemma 2.7, we get that $\widetilde{A}$ is cone-supported at such points. It follows by Lemma 2.9 that the set of points in $\widetilde{A}$, for which there exist farthest points in M, can be covered by countably many Lipschitz hypersurfaces L_n. As Proposition 4.1 shows that we can replace M by $\overline{\mathrm{conv}}M$, we see that $A = \widetilde{A} \setminus \bigcup_n L_n$ satisfies the conclusion of our theorem. ∎

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